EMBEDDINGS OF α-MODULATION SPACES

JOACHIM TOFT AND PATRIK WAHLBERG

Abstract. Abstract. We show upper and lower embeddings of α_1-modulation spaces in α_2-modulation spaces for $0 \le \alpha_1 \le \alpha_2 \le 1$, and prove partial results on the sharpness of the embeddings.

Dedicated to Professor Petar Popivanov on the occasion of his 65th birthday

0. Introduction

Let $1 \le p, q \le \infty$ and define the indices

$$\theta_1(p,q) = \max\left(0, q^{-1} - \min(p^{-1}, p'^{-1})\right),$$
$$\theta_2(p,q) = \min\left(0, q^{-1} - \max(p^{-1}, p'^{-1})\right).$$

Our main result is the following. For $0 \le \alpha_1 \le \alpha_2 \le 1$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, we have the embeddings for α-modulation spaces

$$M^{p,q}_{\alpha_2, s+d(\alpha_2-\alpha_1)\theta_1(p,q)}(\mathbb{R}^d) \subseteq M^{p,q}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{p,q}_{\alpha_2, s+d(\alpha_2-\alpha_1)\theta_2(p,q)}(\mathbb{R}^d). \tag{0.1}$$

(See Theorem 2.3.) The embeddings (0.1) contain known results for embeddings of modulation spaces in Besov spaces [16] and sharpen Gröbner's embeddings [8].

We also show the sharpness of the embeddings (0.1) in the following sense. (See Corollary 3.6.) If $p \ge \min(2, q)$ then

$$M^{p,q}_{\alpha_1,s} \subseteq M^{p,q}_{\alpha_2,t} \quad \Longrightarrow \quad t \le s + d(\alpha_2 - \alpha_1)\theta_2(p,q). \tag{0.2}$$

If $p \le \max(2, q)$ then

$$M^{p,q}_{\alpha_2,t} \subseteq M^{p,q}_{\alpha_1,s} \quad \Longrightarrow \quad t \ge s + d(\alpha_2 - \alpha_1)\theta_1(p,q). \tag{0.3}$$

For $p < \min(2, q)$ we are unable to show the implication (0.2). Nevertheless, we conjecture that the implication (0.2) holds also for $p < \min(2, q)$. By duality, this is equivalent to (0.3) for $p > \max(2, q)$.

Remark 0.1. After finalizing the proof of (0.1), we noticed the preprint [10] by Han and Wang. Their results [10, Theorems 5.1 and 5.2] generalize our Theorem 2.3, and show that the embeddings (0.1) hold for all $p, q \in (0, \infty]$, $0 \le \alpha_1 \le \alpha_2 \le 1$ and $s \in \mathbb{R}$. This paper provides an alternative proof to Han and Wang's proof in the case $p, q \in [1, \infty]$, and establishes the partial sharpness of the embeddings (sharpness results are not treated in [10]).

Key words and phrases. α-modulation spaces, embeddings, sharpness.

1. Preliminaries

$\mathbb{N}_0$ denotes the nonnegative integers. Inclusions $A \subseteq B$ and equalities $A = B$ of topological spaces A, B, are understood as embeddings, that is an inclusion is continuous. We use the standard notations $\mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^d)$, $C_c^\infty(\mathbb{R}^d)$ for function and distribution spaces (see e.g. [11]). The Fourier transform of $f \in \mathscr{S}(\mathbb{R}^d)$ is defined by

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.$$

A Fourier multiplier operator is defined by $\varphi(D)f = \mathscr{F}^{-1}(\varphi \widehat{f})$, provided φ and f are objects such that the expression makes sense. For $s \in \mathbb{R}$ the Sobolev space $H_s(\mathbb{R}^d)$ is defined as the subspace of $f \in \mathscr{S}'(\mathbb{R}^d)$ such that $\widehat{f} \in L^2_{\rm loc}(\mathbb{R}^d)$ and

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

We denote by $|A|$ the cardinality of a finite set A, and by $\mu(A)$ the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}^d$. A closed ball in $\mathbb{R}^d$ of center $a \in \mathbb{R}^d$ and radius $r \geq 0$ is denoted $B(a, r) = \{x \in \mathbb{R}^d : |x - a| \leq r\}$. A closed cube in $\mathbb{R}^d$ of center c and side length $2r$ is denoted $Q(c, r) = \{x \in \mathbb{R}^d : \max_{1 \leq j \leq d} |x_j - c_j| \leq r\}$. The conjugate exponent to $p \in [1, \infty]$ is denoted p' and defined by $1/p + 1/p' = 1$. The notation $X \lesssim Y$ means that $X \leq CY$ for some constant $C > 0$, and $X_i \lesssim Y_j$ for $i \in I$ and $j \in J$ means that the constant is uniformly bounded over the index sets I and J. If $X \lesssim Y$ and $Y \lesssim X$ then we write $X \asymp Y$. Coordinate reflection is denoted $\check{f}(x) = f(-x)$.

1.1. Besov spaces.

Define

$$D_j = \{\xi \in \mathbb{R}^d : 2^{j-2} \leq |\xi| \leq 2^j\}, \quad j \geq 1. \tag{1.1}$$

Let $\{\varphi_j\}_{j=0}^\infty \subseteq C_c^\infty(\mathbb{R}^d)$ be a sequence with the following properties [2].

$$\begin{aligned} &\operatorname{supp} \varphi_0 \subseteq B(0,1), \\ &\operatorname{supp} \varphi_j \subseteq D_j, \quad j \geq 1, \\ &\sum_{j=0}^\infty \varphi_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^d. \end{aligned} \tag{1.2}$$

Then we have for $j \geq 0$

$$2^{j-1} \leq |\xi| \leq 2^j \quad \Rightarrow \quad \varphi_j(\xi) + \varphi_{j+1}(\xi) = 1. \tag{1.3}$$

The functions φ_j for $j \geq 1$ are constructed as dilations $\varphi_j(\xi) = \varphi(2^{1-j}\xi)$ for a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ supported in D_1 (cf. [2]). Let $p, q \in [1, \infty]$

and let $s \in \mathbb{R}$. The Besov space $B_s^{p,q}(\mathbb{R}^d)$ is defined as the space of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{B_s^{p,q}} = \left(\sum_{j=0}^{\infty} \left(2^{js} \|\varphi_j(D) f\|_{L^p} \right)^q \right)^{1/q} < \infty \tag{1.4}$$

when $q < \infty$ and with the standard modification when $q = \infty$ [2]. We abbreviate $B_s^{p,p} = B_s^p$ and $B_0^{p,q} = B^{p,q}$.

1.2. **α-modulation spaces.** We need the following definitions introduced by Feichtinger and Gröbner [4–6, 8] (cf. [3, 7]).

Definition 1.1. A countable set $\mathcal{Q}$ of subsets $Q \subseteq \mathbb{R}^d$ is called an admissible covering provided

$$\bigcup_{Q \in \mathcal{Q}} Q = \mathbb{R}^d,$$

$$|\{Q' \in \mathcal{Q} : Q \cap Q' \neq \emptyset\}| \leq n_0 \quad \forall Q \in \mathcal{Q}, \tag{1.5}$$

for some finite integer n_0.

For each $Q \in \mathcal{Q}$, let

$$r_Q = \sup\{r \in \mathbb{R} : B(c, r) \subseteq Q \text{ for some } c \in \mathbb{R}^d\}, \tag{1.6}$$

$$R_Q = \inf\{R \in \mathbb{R} : Q \subseteq B(c, R) \text{ for some } c \in \mathbb{R}^d\}. \tag{1.7}$$

Definition 1.2. Let $\alpha \in [0, 1]$. An admissible covering $\{Q\}_{Q \in \mathcal{Q}}$ is called an α-covering provided there exists a constant $K \geq 1$ such that

$$\mu(Q) \asymp \langle x \rangle^{\alpha d}, \quad x \in Q, \quad Q \in \mathcal{Q}, \tag{1.8}$$

$$R_Q / r_Q \leq K, \quad Q \in \mathcal{Q}. \tag{1.9}$$

Definition 1.3. Let $\alpha \in [0, 1]$ and let $\{Q\}_{Q \in \mathcal{Q}}$ be an α-covering of $\mathbb{R}^d$. Then $\{\psi_Q\}_{Q \in \mathcal{Q}}$ is called a bounded admissible partition of unity corresponding to $\mathcal{Q}$ ($\mathcal{Q}$-BAPU) provided

$$\operatorname{supp} \psi_Q \subseteq Q, \quad Q \in \mathcal{Q},$$

$$\sum_{Q \in \mathcal{Q}} \psi_Q(\xi) = 1 \quad \forall \xi \in \mathbb{R}^d,$$

$$\sup_{Q \in \mathcal{Q}} \|\mathscr{F} \psi_Q\|_{L^1} < \infty. \tag{1.10}$$

We will call a $\mathcal{Q}$-BAPU an α-BAPU when $\mathcal{Q}$ is an α-covering.

Definition 1.4. Let $\alpha \in [0, 1]$, $p, q \in [1, \infty]$, $s \in \mathbb{R}$, let $\{Q\}_{Q \in \mathcal{Q}}$ be an α-covering of $\mathbb{R}^d$ and let $\{\psi_Q\}_{Q \in \mathcal{Q}}$ be a $\mathcal{Q}$-BAPU. The weighted α-modulation space $M_{\alpha,s}^{p,q}(\mathbb{R}^d)$ is defined as all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M_{\alpha,s}^{p,q}} = \left(\sum_{Q \in \mathcal{Q}} \langle \xi_Q \rangle^{qs} \|\psi_Q(D) f\|_{L^p}^q \right)^{1/q} < \infty \tag{1.11}$$

where $\xi_Q \in Q$ for all $Q \in \mathcal{Q}$, when $q < \infty$. If $q = \infty$ the global l^q norm in (1.11) is replaced by l^∞.

The α-modulation spaces contain as extreme cases the frequency-weighted modulation spaces (cf. [4, 9]) $M_s^{p,q} = M_{0,s}^{p,q}$ ($\alpha = 0$) and the Besov spaces $B_s^{p,q} = M_{1,s}^{p,q}$ ($\alpha = 1$) (cf. [8]). The number α thus parametrizes a scale of spaces that in some sense is intermediate between the modulation spaces and the Besov spaces. We abbreviate $M_{\alpha,s}^{p,p} = M_{\alpha,s}^p$, $M_s^{p,p} = M_s^p$ and $M_0^{p,q} = M^{p,q}$ (the unweighted or classical modulation spaces). For $t \geq s$ we have the embedding $M_{\alpha,t}^{p,q} \subseteq M_{\alpha,s}^{p,q}$, $\alpha \in [0,1]$, $p, q \in [1, \infty]$.

For α in the interval $0 \leq \alpha < 1$, that is, excluding the Besov spaces, we will use the following α-covering and an associated $\mathcal{Q}$-BAPU (cf. [3]). Set

$$B_k = B(k|k|^\beta, r|k|^\beta), \quad k \in \mathbb{Z}^d \setminus 0, \tag{1.12}$$

where $\beta = \alpha/(1-\alpha)$. Note that $B_k = B(\xi_k, r|\xi_k|^\alpha)$ where $\xi_k = k|k|^\beta$. For $r > 0$ sufficiently large, $\mathcal{Q} = \{B_k\}_{k \in \mathbb{Z}^d \setminus 0}$ is an α-covering of $\mathbb{R}^d$ according to [3, Theorem 2.6]. Moreover, a $\mathcal{Q}$-BAPU $\{\psi_k\}_{k \in \mathbb{Z}^d \setminus 0}$ such that $\operatorname{supp} \psi_k \subseteq B_k$ for all $k \in \mathbb{Z}^d \setminus 0$ can be constructed (see [3, Proposition A.1]).

We will use Borup and Nielsen's Banach frame construction for $M_{\alpha,s}^{p,q}(\mathbb{R}^d)$, based on multivariate brushlet systems (cf. [3]). Let

$$Q_k = Q(k|k|^\beta, r|k|^\beta), \quad k \in \mathbb{Z}^d \setminus 0,$$

where again $\beta = \alpha/(1-\alpha)$. If $r > 0$ is sufficiently large then $\mathcal{Q} = \{Q_k\}_{k \in \mathbb{Z}^d \setminus 0}$ is an α-covering of $\mathbb{R}^d$. One can construct a sequence of functions

$$(w_{n,k})_{n \in \mathbb{N}_0^d,\ k \in \mathbb{Z}^d \setminus 0} \subseteq \mathscr{S}(\mathbb{R}^d)$$

such that $(w_{n,k})_{n \in \mathbb{N}_0^d}$ is an orthonormal system, with $\operatorname{supp} \widehat{w}_{n,k} \subseteq Q_k$, for each $k \in \mathbb{Z}^d \setminus 0$. Each function $w_{n,k}$ is constructed as a tensor product

$$w_{n,k} = \bigotimes_{j=1}^d w_{n_j, I_{k,j}} \tag{1.13}$$

where $Q_k = \Pi_{j=1}^d I_{k,j}$, whose components are, simplifying notation to $n = n_j$, $I = I_{k,j}$,

$$w_{n,I}(x) = \sqrt{\frac{\mu(I)}{2}}\, e^{i a_I x} \, \Big(g(\mu(I)(x + e_{n,I})) + g(\mu(I)(x - e_{n,I})) \Big), \quad x \in \mathbb{R},$$

where $e_{n,I} = \pi(n + 1/2)/\mu(I)$, a_I denotes the left end point of I, i.e. $I = [a_I, b_I]$, and $g \in \mathscr{F} C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \widehat{g} \subseteq [0,1]$. For more details about the sequence of functions $(w_{n,k})_{n \in \mathbb{N}_0^d,\ k \in \mathbb{Z}^d \setminus 0}$ we refer to [3].

Borup and Nielsen [3] show that the sequence $(w_{n,k})$ is a Banach frame for $M_{\alpha,s}^{p,q}(\mathbb{R}^d)$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. We restrict our

interest to the exponents $p,q \in [1,\infty]$. Let $p,q \in [1,\infty]$, $s \in \mathbb{R}$, let $f \in M^{p,q}_{\alpha,s}(\mathbb{R}^d)$, and define the coefficient sequence

$$c_{n,k} = (f, w_{n,k})_{L^2}, \quad n \in \mathbb{N}_0^d, \quad k \in \mathbb{Z}^d \setminus 0 \tag{1.14}$$

where $w_{n,k}$ is defined by (1.13). The coefficient operator is defined by $(Df)_{n,k} = c_{n,k}$, $n \in \mathbb{N}_0^d$, $k \in \mathbb{Z}^d \setminus 0$. The Banach frame property means in this case that

$$\|f\|_{M^{p,q}_{\alpha,s}} \asymp \|c\|_{m^{p,q}_{\alpha,s}}, \tag{1.15}$$

where the sequence space $m^{p,q}_{\alpha,s} = m^{p,q}_{\alpha,s}(\mathbb{N}_0^d \times \mathbb{Z}^d \setminus 0)$ is defined by the norm

$$\|c\|_{m^{p,q}_{\alpha,s}} = \left(\sum_{k \in \mathbb{Z}^d \setminus 0} \left(\sum_{n \in \mathbb{N}_0^d} \left(|k|^{\frac{1}{1-\alpha}\left(s+\alpha d\left(\frac{1}{2}-\frac{1}{p}\right)\right)} |c_{n,k}| \right)^p \right)^{q/p} \right)^{1/q} \tag{1.16}$$

when $p,q < \infty$ and suitably modified otherwise. Moreover, there exists a reconstruction operator R defined by

$$R\, c = \sum_{k \in \mathbb{Z}^d \setminus 0,\ n \in \mathbb{N}_0^d} c_{n,k}\, \widetilde{w}_{n,k},$$

where $(\widetilde{w}_{n,k})_{k \in \mathbb{Z}^d \setminus 0, n \in \mathbb{N}_0^d}$ is a dual frame defined by $\widetilde{w}_{n,k} = \psi_k(D) w_{n,k}$, $n \in \mathbb{N}_0^d$, $k \in \mathbb{Z}^d \setminus 0$. The operator R is bounded as

$$\|Rc\|_{M^{p,q}_{\alpha,s}} \lesssim \|c\|_{m^{p,q}_{\alpha,s}}, \quad c \in m^{p,q}_{\alpha,s}, \tag{1.17}$$

and $RD = id_{M^{p,q}_{\alpha,s}}$. These results are proved in [3, Theorem 4.3].

Let $\mathscr{M}^{p,q}_{\alpha,s}(\mathbb{R}^d)$ be the completion of $\mathscr{S}(\mathbb{R}^d)$ in the norm $\|\cdot\|_{M^{p,q}_{\alpha,s}(\mathbb{R}^d)}$. In the next result we collect some important properties of the α-modulation spaces. The result is a generalization of the corresponding result for modulation spaces.

Proposition 1.5. *Let $\alpha \in [0,1]$, $s \in \mathbb{R}$ and $p,q \in [1,\infty]$. The following holds.*

(i) *The space $M^{p,q}_{\alpha,s}(\mathbb{R}^d)$ is a Banach space which is independent of the sequence $\{\xi_Q\}_{Q \in \mathcal{Q}}$ as long as $\xi_Q \in Q$ for all $Q \in \mathcal{Q}$, and also independent of the α-covering $\{Q\}_{Q \in \mathcal{Q}}$ and of the $\mathcal{Q}$-BAPU $\{\psi_Q\}_{Q \in \mathcal{Q}}$. Varying these parameters gives rise to equivalent norms.*

(ii) *The L^2-product $(\cdot,\cdot)$ on $\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ extends to a continuous sesquilinear form on $M^{p,q}_{\alpha,s}(\mathbb{R}^d) \times M^{p',q'}_{\alpha,-s}(\mathbb{R}^d)$. Furthermore,*

$$\|f\| = \sup |(f,g)|$$

with supremum taken over all $g \in \mathscr{S}(\mathbb{R}^d)$ such that $\|g\|_{M^{p',q'}_{\alpha,-s}} \leq 1$, is a norm equivalent to $\|f\|_{M^{p,q}_{\alpha,s}}$. If $p,q < \infty$, then the dual space of $M^{p,q}_{\alpha,s}$ can be identified with $M^{p',q'}_{\alpha,-s}$ through the form $(\cdot,\cdot)$.

(iii) *Assume that* $0 \le \theta \le 1$, $p, q, p_1, p_2, q_1, q_2 \in [1, \infty]$, $s, s_1, s_2 \in \mathbb{R}$ *satisfy*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2.$$

Then complex interpolation gives

$$(\mathscr{M}^{p_1,q_1}_{\alpha,s_1}, \mathscr{M}^{p_2,q_2}_{\alpha,s_2})_{[\theta]} = \mathscr{M}^{p,q}_{\alpha,s}.$$

(iv) *It holds* $\mathscr{M}^{p,q}_{\alpha,s} \subseteq M^{p,q}_{\alpha,s}$ *with equality if* $p < \infty$ *and* $q < \infty$.

Proof. (i) See [5, Theorems 2.2, 2.3 and 3.7] and [6, Theorem 4.1].

(ii) The fact that the dual space of $M^{p,q}_{\alpha,s}$, for $1 \le p, q < \infty$, can be identified with $M^{p',q'}_{\alpha,-s}$ is a consequence of [5, Theorem 2.8]. Let $1 \le p, q \le \infty$. From [5, Theorem 2.3] it follows

$$|(f, g)| \lesssim \|f\|_{M^{p,q}_{\alpha,s}} \|g\|_{M^{p',q'}_{\alpha,-s}}, \quad g \in \mathscr{S}(\mathbb{R}^d).$$

For the reverse inequality we first let $0 \le \alpha < 1$. By (1.15)

$$\|f\|_{M^{p,q}_{\alpha,s}} \lesssim \|c\|_{m^{p,q}_{\alpha,s}},$$

where the sequence c is defined by (1.14). The $m^{p,q}_{\alpha,s}$-norm of c is the mixed $\ell^{p,q}$ norm of ωc, where the weight ω depends on p, α, s as

$$\omega_{n,k} = \omega_k = |k|^{\frac{1}{1-\alpha}\left(s + \alpha d\left(\frac{1}{2} - \frac{1}{p}\right)\right)}.$$

An application of [1, Lemma 3.1] yields

$$\|c\|_{m^{p,q}_{\alpha,s}} = \|\omega c\|_{\ell^{p,q}} = \sup |(\omega c, d)_{\ell^2}|$$

with supremum taken over all sequences $(d_{n,k})$ of finite support and $\|d\|_{\ell^{p',q'}} \le 1$. Let $(d_{n,k})$ be a sequence of finite support such that $\|d\|_{\ell^{p',q'}} \le 1$ and

$$\|\omega c\|_{\ell^{p,q}} \le 2|(\omega c, d)_{\ell^2}|,$$

and set

$$g = \sum_{k \in \mathbb{Z}^d \setminus 0} \sum_{n \in \mathbb{N}_0^d} \omega_k \, d_{n,k} \, w_{n,k}.$$

Then $g \in \mathscr{S}(\mathbb{R}^d)$ since the sum is finite, and $(f, g) = (\omega c, d)_{\ell^2}$. The following inequality follows from the proofs of [3, Lemma 3.2 and Lemma 4.2]. If $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, then

$$\left\| \sum_{k \in \mathbb{Z}^d \setminus 0} \sum_{n \in \mathbb{N}_0^d} d_{n,k} \, w_{n,k} \right\|_{M^{p',q'}_{\alpha,-s}} \lesssim \|d\|_{m^{p',q'}_{\alpha,-s}}.$$

This gives

$$\|g\|_{M^{p',q'}_{\alpha,-s}} \lesssim \|\omega d\|_{m^{p',q'}_{\alpha,-s}} = \|d\|_{\ell^{p',q'}} \le 1.$$

Hence we have proved that $\|f\|_{M^{p,q}_{\alpha,s}} \lesssim \|f\|$ when $0 \le \alpha < 1$.

It remains to prove the corresponding inequality when $\alpha = 1$, in which case $M^{p,q}_{\alpha,s} = B^{p,q}_s$. Let $\{\varphi_j\}_{j=0}^\infty \subseteq C_c^\infty(\mathbb{R}^d)$ be a sequence that

satisfies (1.2) and $\varphi_j(\xi) = \varphi(2^{1-j}\xi)$ for $j \geq 1$ where $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\operatorname{supp}\varphi \subseteq D_1$. The $B_s^{p,q}$-norm defined by (1.4) is the mixed Lebesgue norm $L^{p,q}(\mathbb{R}^d \times \mathbb{N}_0)$, where $\mathbb{R}^d$ is equipped with the Lebesgue measure and $\mathbb{N}_0$ with the counting measure, of the function $F(x,j) = 2^{js}\varphi_j(D)f(x)$. According to [1, Lemma 3.1] we have

$$\|f\|_{B_s^{p,q}} = \sup \left| \sum_{j=0}^{\infty} 2^{js}(\varphi_j(D)f, g_j)_{L^2} \right|$$

where the supremum is taken over all sequences $(g_j)_0^\infty$ of simple functions of compact support g_j such that $g_j \equiv 0$ for $j > N$ for some $N \geq 0$, and

$$\left(\sum_{j=0}^{\infty} \|g_j\|_{L^{p'}}^{q'} \right)^{1/q'} \leq 1$$

if $q' < \infty$, and $\sup_{0 \leq j < \infty} \|g_j\|_{L^{p'}} \leq 1$ if $q' = \infty$. Therefore there exists $N \geq 0$ and $(g_j)_0^N \subseteq L^{p'}(\mathbb{R}^d)$ such that

$$\|f\|_{B_s^{p,q}} \leq 2 \sum_{j=0}^{N} 2^{js}(\varphi_j(D)f, g_j)_{L^2} = 2(f, \sum_{j=0}^{N} 2^{js}\varphi_j(D)g_j)_{L^2}$$

and

$$\left(\sum_{j=0}^{N} \|g_j\|_{L^{p'}}^{q'} \right)^{1/q'} \leq 1 \tag{1.18}$$

(modified as above if $q' = \infty$). Set

$$g = \sum_{j=0}^{N} 2^{js}\varphi_j(D)g_j \in \mathscr{S}(\mathbb{R}^d).$$

We have $\sup_{j \geq 0} \|\mathscr{F}^{-1}\varphi_j\|_{L^1} \lesssim 1$. By means of (1.3) and Young's inequality, we obtain for $k \geq 1$

$$\begin{aligned} \|\varphi_k(D)g\|_{L^{p'}} &= \left\| \sum_{j=k-1}^{\min(N,k+1)} 2^{js}\varphi_k(D)\varphi_j(D)g_j \right\|_{L^{p'}} \\ &\lesssim 2^{(k-1)s}\|g_{k-1}\|_{L^{p'}} + 2^{ks}\|g_k\|_{L^{p'}} + 2^{(k+1)s}\|g_{k+1}\|_{L^{p'}}, \end{aligned}$$

and

$$\begin{aligned} \|\varphi_0(D)g\|_{L^{p'}} &= \left\| \sum_{j=0}^{\min(N,1)} 2^{js}\varphi_0(D)\varphi_j(D)g_j \right\|_{L^{p'}} \\ &\lesssim \|g_0\|_{L^{p'}} + 2^s\|g_1\|_{L^{p'}}, \end{aligned}$$

which gives, by means of (1.18), $\|g\|_{B_{-s}^{p',q'}} \lesssim 1$. It follows that $\|f\|_{M_{s,1}^{p,q}} \lesssim \|f\|$.

(iii) This follows from [5, Corollary 2.4] (cf. [8, Bemerkung F.2]).

(iv) See [5, Theorem 2.2]. □

2. Embeddings of α-modulation spaces

We need the following elementary lemma (cf. [10, Prop. 2.5] and [8]), a proof of which is provided as a service to the reader.

Lemma 2.1. *If $\alpha \in [0,1]$ and $s \in \mathbb{R}$ then $M^2_{\alpha,s}(\mathbb{R}^d) = H_s(\mathbb{R}^d)$.*

Proof. For the Besov space case ($\alpha = 1$) the result $B^2_s(\mathbb{R}^d) = H_s(\mathbb{R}^d)$ is well known (see e.g. [2, Theorem 6.4.4]). Let $0 \le \alpha < 1$. We use the α-covering (1.12) $\{B_k\}_{k\in\mathbb{Z}^d\setminus 0}$ for $r > 0$ sufficiently large, and an associated BAPU $\{\psi_k\}_{k\in\mathbb{Z}^d\setminus 0}$ such that $0 \le \psi_k \le 1$ for all $k \in \mathbb{Z}^d \setminus 0$. Parseval's formula and (1.8) yield

$$\begin{aligned}\|f\|^2_{M^2_{\alpha,s}(\mathbb{R}^d)} &= \sum_{k\in\mathbb{Z}^d\setminus 0} \langle \xi_k\rangle^{2s} \int_{B_k} \psi_k(\xi)^2 |\widehat{f}(\xi)|^2 d\xi \\ &\lesssim \sum_{k\in\mathbb{Z}^d\setminus 0} \int_{B_k} \psi_k(\xi) \langle \xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi = \|f\|^2_{H_s(\mathbb{R}^d)},\end{aligned}$$

i.e. $H_s \subseteq M^2_{\alpha,s}$. For the opposite inclusion, we note that

$$\sum_{k\in\mathbb{Z}^d\setminus 0} \psi_k(\xi)^2 \ge C, \quad \xi \in \mathbb{R}^d, \tag{2.1}$$

holds for some $C > 0$. In fact, if this would not the case, then for any $\varepsilon > 0$ there exists $\xi \in \mathbb{R}^d$ such that

$$\sum_{k\in\mathbb{Z}^d\setminus 0} \psi_k(\xi)^2 < \varepsilon.$$

Let $\varepsilon < n_0^{-2}$ where n_0 is the upper bound (1.5) corresponding to the covering $\{B_k\}_{k\in\mathbb{Z}^d\setminus 0}$, and let $\xi \in \mathbb{R}^d$ denote the corresponding vector. Then $\psi_k(\xi) < \sqrt{\varepsilon}$ for all $k \in \mathbb{Z}^d \setminus 0$. Since $\xi \in B_j$ for some $j \in \mathbb{Z}^d \setminus 0$ we obtain from (1.5)

$$\sum_{k\in\mathbb{Z}^d\setminus 0} \psi_k(\xi) = \sum_{k:\ B_k\cap B_j\neq\emptyset} \psi_k(\xi) < n_0\sqrt{\varepsilon} < 1$$

which is a contradiction. Thus (2.1) holds for some $C > 0$.

By means of (2.1) and again (1.8) we obtain

$$\begin{aligned}\|f\|^2_{H_s(\mathbb{R}^d)} &\le C^{-1} \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d\setminus 0} \psi_k(\xi)^2 \langle \xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \\ &\lesssim \sum_{k\in\mathbb{Z}^d\setminus 0} \langle \xi_k\rangle^{2s} \int_{B_k} \psi_k(\xi)^2 |\widehat{f}(\xi)|^2 d\xi \\ &= \|f\|^2_{M^2_{\alpha,s}(\mathbb{R}^d)},\end{aligned}$$

i.e. $M^2_{\alpha,s} \subseteq H_s$ and the proof is complete. □

Embeddings for α-modulation spaces have been proved by Gröbner [8], Han and Wang [10], and, for the modulation space case $\alpha = 0$, by Okoudjou [13] and the first named author of this article [15, 16].

The result [16, Theorem 2.10] imply the embeddings, for $p, q \in [1, \infty]$ and $s \in \mathbb{R}$,

$$B^{p,q}_{s+d\theta_1(p,q)}(\mathbb{R}^d) \subseteq M^{p,q}_{0,s}(\mathbb{R}^d) \subseteq B^{p,q}_{s+d\theta_2(p,q)}(\mathbb{R}^d). \tag{2.2}$$

Here the indices θ_1 and θ_2 are defined by

$$\begin{aligned} \theta_1(p, q) &= \max\left(0, q^{-1} - \min(p^{-1}, p'^{-1})\right), \\ \theta_2(p, q) &= \min\left(0, q^{-1} - \max(p^{-1}, p'^{-1})\right) = -\theta_1(p', q'). \end{aligned} \tag{2.3}$$

The unweighted versions (i.e. $s = 0$) of these embeddings were proved in [15, Theorem 3.1]. They imply the embeddings, for $p, q \in [1, \infty]$,

$$B^{p,q}_{d\theta_1(p,q)}(\mathbb{R}^d) \subseteq M^{p,q}(\mathbb{R}^d) \subseteq B^{p,q}_{d\theta_2(p,q)}(\mathbb{R}^d), \tag{2.4}$$

and they have been proven to be sharp. The sharpness was obtained independently by Huang and Wang [17, Theorem 1.1], and by Sugimoto and Tomita [14, Theorem 1.2], and means the following. If $p, q \in [1, \infty]$ and $B^{p,q}_s(\mathbb{R}^d) \subseteq M^{p,q}(\mathbb{R}^d)$ then $s \geq d\theta_1(p, q)$. If $p, q \in [1, \infty]$ and $M^{p,q}(\mathbb{R}^d) \subseteq B^{p,q}_s(\mathbb{R}^d)$ then $s \leq d\theta_2(p, q)$. (By duality, the two assertions are equivalent.) This gives the sharpness also for the weighted case (2.2), since $\langle D\rangle^t$ is continuous $B^{p,q}_s \mapsto B^{p,q}_{s-t}$ for any $t, s \in \mathbb{R}$ (cf. [2]) as well as $M^{p,q}_{0,s} \mapsto M^{p,q}_{0,s-t}$ for any $t, s \in \mathbb{R}$ (cf. [16, Cor. 2.3]). The sharpness of (2.2) reads:

$$\begin{aligned} B^{p,q}_t(\mathbb{R}^d) \subseteq M^{p,q}_{0,s}(\mathbb{R}^d) \quad &\Longrightarrow \quad t \geq s + d\theta_1(p, q), \quad p, q \in [1, \infty], \\ M^{p,q}_{0,s}(\mathbb{R}^d) \subseteq B^{p,q}_t(\mathbb{R}^d) \quad &\Longrightarrow \quad t \leq s + d\theta_2(p, q), \quad p, q \in [1, \infty]. \end{aligned}$$

Note that the embeddings (2.2) and (2.4) are restricted to upper and lower embeddings of 0-modulation spaces in 1-modulation spaces, and give no information on upper and lower embeddings of $M^{p,q}_{\alpha_1,s}$ in $M^{p,q}_{\alpha_2,t}$ for general $\alpha_1, \alpha_2 \in [0, 1]$.

Gröbner's embeddings [8, Theorems F.6, F.7 and pp. 66–68] reads

$$M^{p,q}_{\alpha_2,s+d(\alpha_2-\alpha_1)\nu_1(p,q)}(\mathbb{R}^d) \subseteq M^{p,q}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{p,q}_{\alpha_2,s+d(\alpha_2-\alpha_1)\nu_2(p,q)}(\mathbb{R}^d), \tag{2.5}$$

for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, where the indices ν_1 and ν_2 are defined by

$$\begin{aligned} \nu_1(p, q) &= \theta_1(p, q) + \max\left(0, q^{-1} - \max(p^{-1}, p'^{-1})\right), \\ \nu_2(p, q) &= \theta_2(p, q) + \min\left(0, q^{-1} - \min(p^{-1}, p'^{-1})\right) = -\nu_1(p', q'). \end{aligned} \tag{2.6}$$

Since $\nu_1(p, q) \geq \theta_1(p, q)$ and $\nu_2(p, q) \leq \theta_2(p, q)$, the embeddings (2.2) improve Gröbner's embeddings (2.5) when $\alpha_1 = 0$ and $\alpha_2 = 1$.

We are now in a position to present our main embedding theorem, which is both a sharpening of (2.5) and a generalization of (2.2) to

general α-modulation spaces. In the proof of the theorem we need the following lemma.

Lemma 2.2. *Suppose* $0 \le \alpha_1 \le \alpha_2 \le 1$, $\{Q_j\}_{j\in J}$ *is an* α_1*-covering,* $\{P_i\}_{i\in I}$ *is an* α_2*-covering, and let* $\eta_j \in Q_j$ *for all* $j \in J$*, and* $\xi_i \in P_i$ *for all* $i \in I$*. If*

$$\Omega_i = \{\, j \in J\, ;\, Q_j \cap P_i \neq \emptyset \,\}, \quad i \in I,$$
$$\Lambda_j = \{\, i \in I\, ;\, Q_j \cap P_i \neq \emptyset \,\}, \quad j \in J,$$

then

$$|\Omega_i| \lesssim \langle \xi_i \rangle^{d(\alpha_2-\alpha_1)}, \qquad i \in I, \tag{2.7}$$
$$|\Lambda_j| \lesssim 1, \qquad j \in J, \tag{2.8}$$

and $\langle \xi_i\rangle \asymp \langle \eta_j\rangle$ *for* $j \in \Omega_i$ *for all* $i \in I$*, and for* $i \in \Lambda_j$ *for all* $j \in J$*.*

Proof. By the "disjointization lemma" [5, Lemma 2.9], for any admissible covering $\{Q_j\}_{j\in J}$ we can split the index set as $J = \bigcup_{k=1}^{n_0} J_k$, where n_0 is finite, $\{J_k\}$ are pairwise disjoint, and $j, j' \in J_k$, $j \neq j'$ imply $Q_j \cap Q_{j'} = \emptyset$ for $1 \le k \le n_0$.

Let $i \in I$. By (1.8) we have $\mu(Q_j) \asymp \langle \xi_i\rangle^{d\alpha_1}$ for all $j \in \Omega_i$. By (1.7) and (1.9) we have $P_i \subseteq B(c_i, 2R_2)$ where $R_2^d \lesssim \mu(P_i)$, for some $c_i \in \mathbb{R}^d$. Let $j \in \Omega_i$ and $x_j \in Q_j \cap P_i$. Again (1.7), (1.8), (1.9) give $Q_j \subseteq B(b_j, 2R_1)$ where $R_1^d \lesssim \langle x_j\rangle^{d\alpha_1} \lesssim \langle x_j\rangle^{d\alpha_2} \lesssim \mu(P_i) \lesssim R_2^d$, for some $b_j \in \mathbb{R}^d$. It follows that $Q_j \subseteq B(c_i, CR_2)$ for some $C > 0$. Combining these observations, we obtain for $1 \le k \le n_0$

$$\langle \xi_i\rangle^{d\alpha_1} |\Omega_i \cap J_k| \asymp \sum_{j\in\Omega_i\cap J_k} \mu(Q_j) \le \mu(B(c_i, CR_2)) \lesssim \langle \xi_i\rangle^{d\alpha_2},$$

whereupon (2.7) follows from the disjointization lemma. The proof of (2.8) is similar. The final statement of the lemma is a direct consequence of (1.8). □

Theorem 2.3. *Let* $p, q \in [1,\infty]$, $s \in \mathbb{R}$ *and* $0 \le \alpha_1 \le \alpha_2 \le 1$*. Then*

$$M^{p,q}_{\alpha_2, s+d(\alpha_2-\alpha_1)\theta_1(p,q)}(\mathbb{R}^d) \subseteq M^{p,q}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{p,q}_{\alpha_2, s+d(\alpha_2-\alpha_1)\theta_2(p,q)}(\mathbb{R}^d), \tag{2.9}$$

and, for some constant $C > 0$*, it holds for* $f \in \mathscr{S}'(\mathbb{R}^d)$

$$C^{-1}\|f\|_{M^{p,q}_{\alpha_2, s+d(\alpha_2-\alpha_1)\theta_2(p,q)}} \le \|f\|_{M^{p,q}_{\alpha_1,s}} \le C\|f\|_{M^{p,q}_{\alpha_2, s+d(\alpha_2-\alpha_1)\theta_1(p,q)}}.$$

Proof. By duality it suffices to prove the right hand side embedding. Let $s \in \mathbb{R}$, let $\{\varphi_j\}$ be an α_1-BAPU such that $\varphi_j \ge 0$ for all j, let $\{\psi_i\}$ be an α_2-BAPU such that $\psi_i \ge 0$ for all i, let $\eta_j \in \operatorname{supp}\varphi_j$ for all j, and let $\xi_i \in \operatorname{supp}\psi_i$ for all i. If

$$\Omega_i = \{\, j\, ;\, \operatorname{supp}\varphi_j \cap \operatorname{supp}\psi_i \neq \emptyset \,\}$$
$$\Lambda_j = \{\, i\, ;\, \operatorname{supp}\varphi_j \cap \operatorname{supp}\psi_i \neq \emptyset \,\}$$

(2.10)

then by Lemma 2.2

$$\begin{aligned} |\Omega_i| &\lesssim \langle \xi_i \rangle^{d(\alpha_2-\alpha_1)} && \text{for all} \quad i, \\ |\Lambda_j| &\lesssim 1 && \text{for all} \quad j, \end{aligned}$$

and $\langle \xi_i \rangle \asymp \langle \eta_j \rangle$ for $j \in \Omega_i$ for all i, and for $i \in \Lambda_j$ for all j. This gives, using (2.1),

$$\begin{aligned} \|\psi_i(D)f\|_{L^2}^2 \langle \xi_i \rangle^{2s-d(\alpha_2-\alpha_1)} &= \|\psi_i \widehat{f}\|_{L^2}^2 \langle \xi_i \rangle^{2s-d(\alpha_2-\alpha_1)} \\ &\lesssim \sum_{j\in\Omega_i} \int \varphi_j^2(\xi)\psi_i^2(\xi)|\widehat{f}(\xi)|^2 \, d\xi \langle \xi_i \rangle^{2s-d(\alpha_2-\alpha_1)} \\ &\leq \sum_{j\in\Omega_i} \int \varphi_j^2(\xi)|\widehat{f}(\xi)|^2 \, d\xi \langle \xi_i \rangle^{2s-d(\alpha_2-\alpha_1)} \\ &\lesssim \langle \xi_i \rangle^{d(\alpha_2-\alpha_1)} \sup_{j\in\Omega_i} \|\varphi_j \widehat{f}\|_{L^2}^2 \langle \xi_i \rangle^{2s-d(\alpha_2-\alpha_1)} \\ &= \sup_{j\in\Omega_i} \|\varphi_j(D)f\|_{L^2}^2 \langle \eta_j \rangle^{2s}. \end{aligned}$$

Taking the supremum over i we obtain

$$\|f\|_{M^{2,\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)/2}} \lesssim \|f\|_{M^{2,\infty}_{\alpha_1,s}},$$

which proves the embedding

$$M^{2,\infty}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{2,\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)/2}(\mathbb{R}^d). \tag{2.11}$$

Next we observe that Young's inequality and (1.10) for $\{\psi_i\}$ gives, for all i and any $p \in [1,\infty]$,

$$\|\psi_i(D)f\|_{L^p} = \left\| \sum_{j\in\Omega_i} \mathscr{F}^{-1}\left(\psi_i\varphi_j\widehat{f}\right) \right\|_{L^p} \lesssim \sum_{j\in\Omega_i} \|\varphi_j(D)f\|_{L^p}. \tag{2.12}$$

This gives

$$\begin{aligned} \|f\|_{M^1_{\alpha_2,s}} &= \sum_i \langle \xi_i \rangle^s \|\psi_i(D)f\|_{L^1} \lesssim \sum_i \sum_{j\in\Omega_i} \langle \xi_i \rangle^s \|\varphi_j(D)f\|_{L^1} \\ &\asymp \sum_i \sum_{j\in\Omega_i} \langle \eta_j \rangle^s \|\varphi_j(D)f\|_{L^1} = \sum_j \sum_{i\in\Lambda_j} \langle \eta_j \rangle^s \|\varphi_j(D)f\|_{L^1} \\ &\lesssim \|f\|_{M^1_{\alpha_1,s}}, \end{aligned}$$

which proves the embedding

$$M^1_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^1_{\alpha_2,s}(\mathbb{R}^d). \tag{2.13}$$

We also obtain from (2.12)

$$\begin{aligned} \|f\|_{M^{1,\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)}} &= \sup_i \langle \xi_i \rangle^{s-d(\alpha_2-\alpha_1)} \|\psi_i(D)f\|_{L^1} \\ &\lesssim \sup_i \sum_{j\in\Omega_i} \langle \xi_i \rangle^{-d(\alpha_2-\alpha_1)} \langle \eta_j \rangle^s \|\varphi_j(D)f\|_{L^1} \lesssim \|f\|_{M^{1,\infty}_{\alpha_1,s}}, \end{aligned}$$

which proves the embedding

$$M^{1,\infty}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{1,\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)}(\mathbb{R}^d). \tag{2.14}$$

Again (2.12) gives

$$\begin{aligned}\|f\|_{M^{\infty,1}_{\alpha_2,s}} &= \sum_i \langle \xi_i\rangle^s \|\psi_i(D)f\|_{L^\infty} \lesssim \sum_i \sum_{j\in\Omega_i} \langle \eta_j\rangle^s \left\|\varphi_j(D)f\right\|_{L^\infty} \\ &= \sum_j \sum_{i\in\Lambda_i} \langle \eta_j\rangle^s \left\|\varphi_j(D)f\right\|_{L^\infty} \lesssim \|f\|_{M^{\infty,1}_{\alpha_1,s}},\end{aligned}$$

which proves the embedding

$$M^{\infty,1}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{\infty,1}_{\alpha_2,s}(\mathbb{R}^d). \tag{2.15}$$

Finally (2.12) gives

$$\begin{aligned}\|f\|_{M^{\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)}} &= \sup_i \langle \xi_i\rangle^{s-d(\alpha_2-\alpha_1)} \|\psi_i(D)f\|_{L^\infty} \\ &\lesssim \sup_i \sum_{j\in\Omega_i} \langle \xi_i\rangle^{-d(\alpha_2-\alpha_1)} \langle \eta_j\rangle^s \left\|\varphi_j(D)f\right\|_{L^\infty} \\ &\lesssim \|f\|_{M^{\infty}_{\alpha_1,s}},\end{aligned}$$

which proves the embedding

$$M^{\infty}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)}(\mathbb{R}^d). \tag{2.16}$$

By Lemma 2.1 we have

$$M^{2}_{\alpha_1,s}(\mathbb{R}^d) = M^{2}_{\alpha_2,s}(\mathbb{R}^d). \tag{2.17}$$

The result now follws from interpolation between (2.11), (2.13), (2.14), (2.15), (2.16) and (2.17), and duality. □

3. Sharpness of the embeddings

The notion of α-covering is connected with the metric calculus presented in [12, Section 18.4]. Let $0 \le \alpha \le 1$, and let g be the Riemannian metric

$$g_\eta(\xi) = \frac{|\xi|^2}{\langle \eta\rangle^{2\alpha}}.$$

If $0 < r < 1$ then it follows by straight-forward considerations that

$$g_\eta(\xi-\eta) \le r^2 \quad \Longrightarrow \quad C^{-1} g_\eta(\zeta) \le g_\xi(\zeta) \le C g_\eta(\zeta), \quad \zeta \in \mathbb{R}^d,$$

for some constant C which depends on r only. Hence g is a slowly varying metric in the sense of [12, Def. 18.4.1], and (18.4.2) in [12] is satisfied with $c = r^2$. The results in [12] gives the following proposition.

Proposition 3.1. *Let $0 \le \alpha \le 1$ and $0 < r < 1$. The following holds.*

(i) *For some sequence $\{\xi_i\}_{i\in I} \subseteq \mathbb{R}^d$, the balls $B_i = B(\xi_i, r\langle \xi_i\rangle^\alpha/2)$ constitute an α-covering.*

(ii) *There are functions* $\psi_i \in C_c^\infty(\mathbb{R}^d)$, $i \in I$, *such that* $\operatorname{supp} \psi_i \subseteq B_i$, $0 \le \psi_i \le 1$, $\sum_{i \in I} \psi_i = 1$, *and for every multiindex* β, *there is a finite constant* $C_\beta > 0$ *such that*

$$\sup_{i \in I} \Big(\langle \xi_i \rangle^{\alpha|\beta|} \| \partial^\beta \psi_i \|_{L^\infty} \Big) \le C_\beta. \tag{3.1}$$

(iii) *If* $\mathcal{Q} = \{B_i\}_{i \in I}$ *then* $\{\psi_i\}_{i \in I}$ *is a* $\mathcal{Q}$*-BAPU.*

Proof. (i) and (ii) follow immediately from [12, Lemma 18.4.4] with $\varepsilon < 1/8$. Therefore, in order to prove (iii) it suffices to show

$$\sup_{i \in I} \| \mathscr{F} \psi_i \|_{L^1} < \infty,$$

which is a special case of the following Lemma 3.2. □

Lemma 3.2. *Let* $0 \le \alpha \le 1$ *and suppose* $\{\psi_i\}_{i \in I} \subseteq C_c^\infty(\mathbb{R}^d)$ *is a family of functions such that* $\operatorname{supp} \psi_i \subseteq B(\xi_i, r\langle \xi_i \rangle^\alpha)$, $i \in I$, *for some sequence* $\{\xi_i\}_{i \in I} \subseteq \mathbb{R}^d$ *and some* $r > 0$, *and for any multiindex* β *there is* $C_\beta > 0$ *such that*

$$\sup_{i \in I} \Big(\langle \xi_i \rangle^{\alpha|\beta|} \| \partial^\beta \psi_i \|_{L^\infty} \Big) \le C_\beta. \tag{3.2}$$

Then for $p \in [1, \infty]$ *there is a constant* $C_p > 0$ *such that*

$$\sup_{i \in I} \langle \xi_i \rangle^{-d\alpha/p'} \| \mathscr{F} \psi_i \|_{L^p} \le C_p.$$

Proof. Set

$$\varphi_i(\xi) = \psi_i(\langle \xi_i \rangle^\alpha \xi + \xi_i), \quad i \in I.$$

Then $\operatorname{supp} \varphi_i \subseteq B(0, r)$ for all $i \in I$, and (3.2) gives $\| \partial^\beta \varphi_i \|_{L^\infty} \le C_\beta$ for all $i \in I$. If $p < \infty$ and $n > d/(2p)$ is an integer then integration by parts gives, for some constants c_β,

$$\begin{aligned}
\| \mathscr{F} \varphi_i \|_{L^p}^p &= (2\pi)^{-dp/2} \int_{\mathbb{R}^d} \langle x \rangle^{-2np} \left| \int_{\mathbb{R}^d} \varphi_i(\xi) \langle x \rangle^{2n} e^{-ix\cdot\xi} d\xi \right|^p dx \\
&= (2\pi)^{-dp/2} \int_{\mathbb{R}^d} \langle x \rangle^{-2np} \left| \sum_{|\beta| \le 2n} c_\beta \int_{\mathbb{R}^d} \partial^\beta \varphi_i(\xi) e^{-ix\cdot\xi} d\xi \right|^p dx \\
&\lesssim \int_{\mathbb{R}^d} \langle x \rangle^{-2np} \left(\sum_{|\beta| \le 2n} \| \partial^\beta \varphi_i \|_{L^1} \right)^p dx \lesssim 1
\end{aligned}$$

for all $i \in I$. If $p = \infty$ the observations above give $\| \mathscr{F} \varphi_i \|_{L^\infty} \le (2\pi)^{-d/2} \| \varphi_i \|_{L^1} \lesssim 1$ for all $i \in I$. The result now follows from $\| \mathscr{F} \psi_i \|_{L^p} = \langle \xi_i \rangle^{d\alpha/p'} \| \mathscr{F} \varphi_i \|_{L^p}$. □

Given an α-covering and an α-BAPU according to Proposition 3.1, the next lemma says that we may adjoin a sequence of balls to the covering, and modify the BAPU accordingly, without destroying the α-covering and the α-BAPU properties. A function indexed by the new

index set equals one on a ball of radius proportional to $\langle \xi_j \rangle^\alpha$ where ξ_j is the center of the support of the function. This will be useful in the proofs of the forthcoming sharpness results Propositions 3.4 and 3.5.

Lemma 3.3. *Let $0 \le \alpha \le 1$, $0 < r < 1$, and let $\{B_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ be as in Proposition 3.1. Let J be a countable index set such that $I \cap J = \emptyset$, and let $\{B_j\}_{j \in J}$ be balls such that $B_j = B(\xi_j, r\langle \xi_j \rangle^\alpha/2)$ where $\xi_j \in \mathbb{R}^d$ for $j \in J$, and $B_j \cap B_k = \emptyset$, when $j, k \in J$ and $j \neq k$.*

Then there are functions $\varphi_i \in C_c^\infty(\mathbb{R}^d)$, $i \in I \cup J$, such that the following is true:

(i) $0 \le \varphi_i \le 1$, $\operatorname{supp} \varphi_i \subseteq B_i$ *when* $i \in I \cup J$;

(ii) $\varphi_j = 1$ *on* $B(\xi_j, r\langle \xi_j \rangle^\alpha/4)$ *for* $j \in J$;

(iii) $\{\varphi_i\}_{i \in I \cup J}$ *is an α-BAPU, and for each multiindex β there exists $C_\beta > 0$ such that*

$$\sup_{i \in I \cup J} \Big(\langle \xi_i \rangle^{\alpha|\beta|} \|\partial^\beta \varphi_i\|_{L^\infty} \Big) \le C_\beta. \tag{3.3}$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $0 \le \varphi \le 1$, $\operatorname{supp} \varphi \subseteq B(0, r/2)$ and $\varphi(\xi) = 1$ for $\xi \in B(0, r/4)$. We set

$$\varphi_j(\xi) = \varphi(\langle \xi_j \rangle^{-\alpha}(\xi - \xi_j)) \qquad \text{for} \quad j \in J$$

and

$$\varphi_i(\xi) = \psi_i(\xi) \prod_{j \in J} (1 - \varphi_j(\xi)) \quad \text{for} \quad i \in I.$$

Then properties (i) and (ii) are satisfied. The estimate $\sup_{j \in J} \langle \xi_j \rangle^{\alpha|\beta|} \|\partial^\beta \varphi_j\|_{L^\infty} < C_\beta$ for any multiindex β follows immediately. These estimates combined with (3.1) and straightforward considerations give $\sup_{i \in I} \langle \xi_i \rangle^{\alpha|\beta|} \|\partial^\beta \varphi_i\|_{L^\infty} < C_\beta$ for all multiindices β. Thus (3.3) holds for all multiindices β. Likewise one can easily verify

$$\sum_{i \in I \cup J} \varphi_i(\xi) = 1 \quad \forall \xi \in \mathbb{R}^d,$$

as well as the fact that $\{B_i, B_j\}_{i \in I, j \in J}$ is an admissible α-covering. To prove (iii) it thus suffices to observe that $\sup_{j \in J} \|\mathscr{F} \varphi_j\|_{L^1} < \infty$ follows from $\|\mathscr{F} \varphi_j\|_{L^1} = \|\mathscr{F} \varphi\|_{L^1}$, and that $\sup_{i \in I} \|\mathscr{F} \varphi_i\|_{L^1} < \infty$ follows from (3.3) and Lemma 3.2. $\square$

We are now in a position to prove two results which show that the embeddings (2.9) in Theorem 2.3 are optimal, in most cases. This is a consequence of the following Propositions 3.4 and 3.5.

Proposition 3.4. *If $p, q \in [1, \infty]$, $0 \le \alpha_1 \le \alpha_2 \le 1$ and $t, s \in \mathbb{R}$ then*

$$M^{p,q}_{\alpha_1,s} \subseteq M^{p,q}_{\alpha_2,t} \quad \Longrightarrow \quad t \le s + d(\alpha_2 - \alpha_1)\Big(\frac{1}{q} - \frac{1}{p'}\Big).$$

Proof. We prove the result by showing that the assumption

$$\varepsilon := t - s - d(\alpha_2 - \alpha_1)(1/q - 1/p') > 0$$

implies that

$$M^{p,q}_{\alpha_1,s} \subseteq M^{p,q}_{\alpha_2,t} \tag{3.4}$$

cannot hold.

Let $\{\varphi_j\}_{j\in J}$ be an α_1-BAPU constructed according to Proposition 3.1, and let $\{\psi_i\}$ be an α_2-BAPU constructed according to Proposition 3.1 and modified according to Lemma 3.3. Then there exists an infinite index set I such that the following is true for some $r > 0$:

(i) If $i_1, i_2 \in I$ and $i_1 \neq i_2$, then $\operatorname{supp}\psi_{i_1} \cap \operatorname{supp}\psi_{i_2} = \emptyset$;
(ii) $\psi_i(\xi) = 1$ on $B_i = B(\xi_i, r\langle\xi_i\rangle^{\alpha_2})$, $\xi_i \in \mathbb{R}^d$, $i \in I$.

Let $\vartheta \in C_c^\infty(\mathbb{R}^d)$ satisfy $0 \le \vartheta \le 1$, $\operatorname{supp}\vartheta \subseteq B(0,r)$ and $\vartheta(\xi) = 1$ when $\xi \in B(0, r/2)$, and define $\vartheta_i(\xi) = \vartheta(\langle\xi_i\rangle^{-\alpha_2}(\xi - \xi_i))$. Then $\psi_i = 1$ in $\operatorname{supp}\vartheta_i$. Let $I' \subseteq I$ be any finite subset, let $\{t_i\}_{i\in I'}$ be a sequence of nonnegative numbers, and set

$$\widehat{f}(\xi) = \sum_{i\in I'} t_i\vartheta_i(\xi) \in C_c^\infty(\mathbb{R}^d).$$

Let $q < \infty$. It follows from our choice of ϑ_i that

$$\begin{aligned}\|f\|_{M^{p,q}_{\alpha_2,t}} &\ge \Big(\sum_{i\in I'} \big(\langle\xi_i\rangle^t \|\psi_i(D)f\|_{L^p}\big)^q\Big)^{1/q} \\ &= \Big(\sum_{i\in I'} \big(\langle\xi_i\rangle^t t_i\, \|\widehat{\vartheta}_i\|_{L^p}\big)^q\Big)^{1/q} \asymp \Big(\sum_{i\in I'} \big(t_i \langle\xi_i\rangle^{t+d\alpha_2/p'}\big)^q\Big)^{1/q}.\end{aligned} \tag{3.5}$$

Next we estimate $\|f\|_{M^{p,q}_{\alpha_1,s}}$. Set

$$\begin{aligned} J_i &= \{\, j \in J\,;\, \operatorname{supp}\varphi_j \cap B_i \neq \emptyset \,\}, \quad i \in I', \\ I'_j &= \{\, i \in I'\,;\, \operatorname{supp}\varphi_j \cap B_i \neq \emptyset \,\}, \quad j \in J. \end{aligned}$$

By Lemma 2.2,

$$\begin{aligned} |J_i| &\lesssim \langle\xi_i\rangle^{d(\alpha_2-\alpha_1)}, && i \in I', \\ |I'_j| &\lesssim 1, && j \in J. \end{aligned}$$

Denoting the center of the ball in which φ_j is supported by $\eta_j \in \mathbb{R}^d$, this gives, using Hölder's and Young's inequalities, Lemma 2.2 and Lemma

3.2,

$$
\begin{aligned}
\|f\|_{M^{p,q}_{\alpha_1,s}} &= \left(\sum_{j\in J} \langle \eta_j\rangle^{sq} \left\| \sum_{i\in I'_j} t_i \mathscr{F}^{-1}\left(\varphi_j \vartheta_i\right) \right\|_{L^p}^q \right)^{1/q} \\
&\lesssim \left(\sum_{j\in J} \langle \eta_j\rangle^{sq} \sum_{i\in I'_j} t_i^q \|\mathscr{F}^{-1}\left(\varphi_j \vartheta_i\right)\|_{L^p}^q \right)^{1/q} \\
&\lesssim \left(\sum_{i\in I'} \sum_{j\in J_i} \langle \eta_j\rangle^{sq} t_i^q \|\mathscr{F}^{-1}\vartheta_i\|_{L^1}^q \|\mathscr{F}^{-1}\varphi_j\|_{L^p}^q \right)^{1/q} \\
&\lesssim \left(\sum_{i\in I'} \sum_{j\in J_i} \langle \eta_j\rangle^{sq} t_i^q \|\mathscr{F}^{-1}\varphi_j\|_{L^p}^q \right)^{1/q} \\
&\lesssim \left(\sum_{i\in I'} \sum_{j\in J_i} \langle \xi_i\rangle^{sq+d\alpha_1 q/p'} t_i^q \right)^{1/q} \\
&\lesssim \left(\sum_{i\in I'} \left(t_i \langle \xi_i\rangle^{s+d(\alpha_2-\alpha_1)/q+d\alpha_1/p'} \right)^q \right)^{1/q} .
\end{aligned} \tag{3.6}
$$

We may assume that $I = \mathbb{N}_0$. Since $|\xi_i| \to \infty$ as $i \to \infty$, we may assume that $\langle \xi_i\rangle \ge \langle i\rangle^{\frac{2}{\varepsilon q}}$, by passing to a subsequence if necessary. If we set

$$t_i := \langle i\rangle^{-\frac{2}{q}} \langle \xi_i\rangle^{-s-d(\alpha_2-\alpha_1)/q-d\alpha_1/p'}$$

then (3.5) and (3.6) give a contradiction to (3.4), as $|I'|$ is made arbitrarily large. This proves the result when $q<\infty$. The case $q=\infty$ is settled with slight modifications of the same proof. □

Proposition 3.5. *If $p,q\in[1,\infty]$, $0\le\alpha_1\le\alpha_2\le 1$ and $t,s\in\mathbb{R}$ then*

$$M^{p,q}_{\alpha_1,s} \subseteq M^{p,q}_{\alpha_2,t} \quad \Longrightarrow \quad t \le s.$$

Proof. We show that $t>s$ implies that (3.4) does not hold.

Let $\{\varphi_j\}_{j\in J}$, $\{\psi_i\}$ and I be as in the proof of Proposition 3.4 and let $\vartheta_i = \vartheta(\xi-\xi_i) \in C_c^\infty(\mathbb{R}^d)$, where $\vartheta\in C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp}\vartheta \subseteq B(0,r)$ is the same as in the proof of Proposition 3.4. Let f be given by

$$\widehat{f}(\xi) = \sum_{i\in I'} t_i \vartheta_i(\xi) \in C_c^\infty(\mathbb{R}^d)$$

for some suitable sequence $\{t_i\}_{i\in I'}$ where $I'\subseteq I$ is finite. Let $q<\infty$. We have

$$
\begin{aligned}
(3.7)\quad \|f\|_{M^{p,q}_{\alpha_2,t}} &\ge \Big(\sum_{i\in I'}\big(\langle\xi_i\rangle^t\|\psi_i(D)f\|_{L^p}\big)^q\Big)^{1/q}\\
&=\Big(\sum_{i\in I'}\big(\langle\xi_i\rangle^t t_i\,\|\widehat{\vartheta}_i\|_{L^p}\big)^q\Big)^{1/q}\asymp\Big(\sum_{i\in I'}\big(t_i\langle\xi_i\rangle^t\big)^q\Big)^{1/q}.
\end{aligned}
$$

In order to estimate $\|f\|_{M^{p,q}_{\alpha_1,s}}$ we set

$$
\begin{aligned}
J_i &= \{\, j\in J\,;\, \operatorname{supp}\varphi_j\cap B(\xi_i,r)\neq\emptyset\,\},\quad i\in I',\\
I'_j &= \{\, i\in I'\,;\, \operatorname{supp}\varphi_j\cap B(\xi_i,r)\neq\emptyset\,\},\quad j\in J.
\end{aligned}
$$

As in the proof of Lemma 2.2 it follows that

$$
\sup_{i\in I'}|J_i|<\infty,\quad \sup_{j\in J}|I'_j|<\infty,\quad\text{and}\quad \langle\xi_i\rangle\asymp\langle\eta_j\rangle\quad\text{when}\quad j\in J_i.
$$

As in the estimate (3.6) this gives, again using Hölder's and Young's inequalities and Lemma 3.2,

$$
\begin{aligned}
\|f\|_{M^{p,q}_{\alpha_1,s}} &= \left(\sum_{j\in J}\langle\eta_j\rangle^{sq}\left\|\sum_{i\in I'_j}t_i\mathscr{F}^{-1}\left(\varphi_j\vartheta_i\right)\right\|^q_{L^p}\right)^{1/q}\\
&\lesssim\left(\sum_{j\in J}\langle\eta_j\rangle^{sq}\sum_{i\in I'_j}t_i^q\|\mathscr{F}^{-1}\left(\varphi_j\vartheta_i\right)\|^q_{L^p}\right)^{1/q}\\
&\lesssim\left(\sum_{i\in I'}\sum_{j\in J_i}\langle\xi_i\rangle^{sq}t_i^q\|\mathscr{F}^{-1}\vartheta_i\|^q_{L^p}\|\mathscr{F}^{-1}\varphi_j\|^q_{L^1}\right)^{1/q}\\
&\lesssim\left(\sum_{i\in I'}\langle\xi_i\rangle^{sq}t_i^q\right)^{1/q}.
\end{aligned}\tag{3.8}
$$

As before (3.7) and (3.8) give a contradiction to (3.4). The case $q=\infty$ follows in the same manner. $\square$

A combination of (2.3), Propositions 3.4 and 3.5, and duality give the earlier mentioned optimality result concerning Theorem 2.3.

Corollary 3.6. *Let $p,q\in[1,\infty]$, $s\in\mathbb{R}$ and $0\le\alpha_1\le\alpha_2\le 1$. If $1/p\le\max(1/2,1/q)$ then*

$$
M^{p,q}_{\alpha_1,s}\subseteq M^{p,q}_{\alpha_2,t}\quad\Longrightarrow\quad t\le s+d(\alpha_2-\alpha_1)\theta_2(p,q).
$$

If $1/p\ge\min(1/2,1/q)$ then

$$
M^{p,q}_{\alpha_2,t}\subseteq M^{p,q}_{\alpha_1,s}\quad\Longrightarrow\quad t\ge s+d(\alpha_2-\alpha_1)\theta_1(p,q).
$$

Acknowledgements. Part of the content of this article was presented at the International Conference on Partial Differential Equations and Applications, Sofia, Bulgaria, September 14–17, 2011, in honor of Professor Petar Popivanov on the occasion of his 65th anniversary. The second named author thanks the organizers of the conference Georgi Boyadjiev, Todor Gramchev, Nikolay Kutev and Alessandro Oliaro, for the interesting conference, and for the invitation to give a talk.

Furthermore, an early version of the content was also presented at the conference From Abstract to Computational Harmonic Analysis, June 13–19, 2011, Strobl, Austria. We thank the organizers Karlheinz Gröchenig and Thomas Strohmer, as well as Hans G. Feichtinger, for the opportunity to present the paper and for the stimulating conference.

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School of Computer Science, Physics and Mathematics, Linnæus University, SE–351 95 Växjö, Sweden.
E-mail address: `joachim.toft@lnu.se`

Department of Mathematics, University of Turin, Via Carlo Alberto 10, 10123 Torino (TO), Italy.
E-mail address: `patrik.wahlberg@unito.it`